MATH 140A Review: proof by contradiction and the contrapositive

1. Show that $\sqrt{5}+\sqrt{7}$ is irrational.

Solution:

Proof. By way of contradiction, we assume that $\sqrt{5}+\sqrt{7}$ is rational. By definition, we have $\sqrt{5}+\sqrt{7}=p/q$ for some $p,q\in\mathbb{Z}$ where $q\neq 0$. Let us look for a contradiction. We have

$$5-7=(\sqrt{5}+\sqrt{7})(\sqrt{5}-\sqrt{7})=p/q(\sqrt{5}-\sqrt{7}).$$

Thus,

$$-2q/p=\sqrt{5}-\sqrt{7}.$$

Adding this to $\sqrt{5}+\sqrt{7}=p/q$ gives us:

$$-2q/p+p/q=(\sqrt{5}-\sqrt{7})+(\sqrt{5}+\sqrt{7})=2\sqrt{5}.$$

Thus,

$$(-2q/p+p/q)/2=\sqrt{5}.$$

Since we know that the rationals are closed under multiplication and addition, then $(-2q/p+p/q)/2=\sqrt{5}$ is rational. This is a contradiction because we know that $\sqrt{5}$ is irrational. Thus, $\sqrt{5}+\sqrt{7}$ is irrational.

□

2. Assume that x is an integer. If x^7-3x^5+88 is odd, then x is odd.

Solution:

Proof. We will show the contrapositive: If x is even, then x^7-3x^5+88 is even.

Assume x is even. Then x^7 and $-3x^5$ are both even. Therefore, x^7-3x^5 is even. Hence, x^7-3x^5+88 an even number. Since the contrapositive statement is true, then the original statement is true.

□

3. Find the contrapositive of: Let a_n be a sequence of real numbers. If $\sum_{n=0}^{\infty} a_n$ converges, then $a_n\to 0$ as $n\to\infty$.

Solution: The contrapositive is: If $\lim_{n\to} a_n$ does not exist or does not converge to 0, then $\sum_{n=0}^{\infty} a_n$ does not converge.